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Pair production and vacuum polarization of vector particles with electric dipole moments and anomalous magnetic moments

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Abstract. The matrix 8-component Dirac-like form of the P-odd equations for boson fields of spin 1 and 0 are obtained and the $GL(2,c)$ symmetry group of the equations is derived. We found exact solutions of the field equation for vector particles with arbitrary electric and magnetic moments in external constant and uniform electromagnetic fields. The differential probability of pair production of vector particles with electric dipole moments and anomalous magnetic moments by an external constant and uniform electromagnetic field has been found using exact solutions. We have calculated the imaginary and real parts of the electromagnetic field Lagrangian that takes into account the vacuum polarization of vector particles.

1 Introduction

The vector particles $W^{\pm}$, Z^0 play very important roles as carriers of the weak interactions. The standard model of the electroweak interactions (SM), which implies the Higgs mechanism of acquiring mass of the vector particles, is renormalizable. Renormalizable electrodynamics for massive charged vector bosons is based in the framework of the SM on the spontaneous breaking of the local $SU(2)_{\mathrm{L}} \otimes U(1)$-symmetry. The $U(1)$ subgroup is unbroken and the corresponding gauge electromagnetic field remains massless. At the same time the gauge fields, which are identified with the intermediate vector bosons ($W^{\pm}$, Z^0) corresponding to the broken $SU(2)_{\mathrm{L}}$ subgroup, acquire masses. There is a certain symmetry of the vector electromagnetic vertices in the renormalizable SM and as a result the gyromagnetic ratio for vector particles is equal to two. It should be noticed that for the non-renormalizable case of the Proca Lagrangian the gyromagnetic ratio $g = 1$. So if an anomalous magnetic moment (AMM) of the vector particles is observed which corresponds to $g \neq 2$, that would signal physics beyond the SM.

The CP violation observed in the decays of the K^0-mesons and in $B_d^0/\overline{B}_d^0 \to J/\psi K_{\mathrm{s}}^0$ decays remains mysterious. In the SM, CP-violating interactions can be explained by the Kobayashi–Maskawa mechanism which presupposes a single phase for three quark generations. In this scheme the predicted electric dipole moments (EDM's) of the elementary particles are extremely small. In some supersymmetric and multi-Higgs models which are extensions of the SM, CP-violating effects are much stronger [1]. The EDM of particles violates the time-reversal (T) symmetry and the CP invariance, which are equivalent due to the CPT-theorem [2]. Some aspects of the CP-violating effects which follow from the EDM of the neutron, electron and atoms are discussed in [3]. The EDM bounds of the neutron and the electron can be established in low energy experiments.

There are some investigations of the EDM of vector W-bosons in the framework of the SM and beyond in [4]. But for the W-bosons it is necessary to analyze the high energy processes for extracting CP-odd asymmetries. The EDM of the W-bosons can give a large contribution to the EDM of fermions (in particular to electrons). The EDM of particles may be also induced by Higgs-boson exchange [5]. The prediction of the EDM of the W-boson in the SM is $d_W \simeq 10^{-29}\,e\,\mathrm{cm}$ [6] but beyond the SM it can be (for example in the two-Higgs-doublet model) about 10^{-21}–$10^{-20}\,e\,\mathrm{cm}$ (see last reference in [4]). The experimental constraint on the EDM of the W-boson which follows from the experimental upper bound for the neutron EDM $d_n = (-3 \pm 5) \times 10^{-26}\,e\,\mathrm{cm}$ [7] is $d_W \leq 10^{-19}\,e\,\mathrm{cm}$. So the presence of the EDM can indicate physics beyond the SM.

The strong interacting composite hadrons ρ, ω (and others) possess spin one. The theory of strong interactions of quarks and gluons, quantum chromodynamics (QCD), is renormalizable. However, the properties of hadrons are described by the infrared region of QCD where perturbation theory in the small parameter α_{s} is not acceptable. In this region some phenomenological models are used. A non-perturbative theory of the strong interactions of hadrons has not been developed yet, but there is progress in describing hadrons in the framework of QCD string theory [8]. In this approach the EDM of mesons [9] and baryons [10] appears naturally. It should be noticed that the EDM of the neutron may be induced by the ϑ-term of the QCD vacuum. The QCD vacuum angle ϑ violates P and CP symmetries and gives CP-odd electromagnetic observables. As the EDM of the neutron is small the ϑ-

parameter of the QCD vacuum is also small. It is possible to explore the axion mechanism [11] to solve the strong CP problem of having the parameter $\vartheta = 0$. Vector mesons may possess an EDM due to the ϑ-term [12] and CP-odd electromagnetic form-factors of the ρ-mesons can be introduced.

In view of the great interest to physics in the framework of the SM, and physics beyond the SM, it is very important to study the various processes involving massive vector bosons with the EDM and AMM. The important and interesting vacuum quantum effects are pair production of particles and antiparticles and vacuum polarization [13]. In particular, there is a vacuum instability of the vector particles in a magnetic field [14]. This is due to the large contribution of the tachyon mode to the negative part of the Callan–Symanzik β-function, and as a result the vacuum is reconstructed in a magnetic field. Some studies were performed to investigate the vacuum quantum effects for vector fields. Pair production and vacuum polarization of vector fields with a gyromagnetic ratio $g = 2$ by a constant uniform electric field were investigated in [15]. The semiclassical imaginary-time method was used in [16] to find the probability of pair production by a constant electromagnetic field for arbitrary spin s and gyromagnetic ratio g. In [17] we found the pair production probability and the vacuum polarization of fields for arbitrary s and g on the basis of the exact solutions of the wave equation for particles in a constant and uniform electromagnetic field and with the help of the Fock–Schwinger proper-time method. In this approach fields realize the $(s, 0) \oplus (0, s)$-representation of the Lorentz group. Nonlinear corrections to the constant uniform electromagnetic field due to the vacuum polarization of a charged vector field in the framework of the renormalizable gauge theory were studied in [18]. The pair production probability of charged vector bosons with $g = 1$ by a non-stationary electric field was derived in [19].

In this work we study the pair production probability and the vacuum polarization of the charged vector particles with arbitrary EDM and AMM. This paper is organized as follows. In Sect. 2 we proceed from the Dirac–Kähler equations for boson fields of spin one and zero. We show that this system of wave equations can be represented as two subsystems of P-odd equations for self-dual and antiself-dual antisymmetric tensors of second rank. The matrix form and the symmetry group of the equations is investigated in Sect. 3. In Sect. 4 a P-odd system of first order equations for vector fields with the EDM and AMM is introduced. We found exact solutions of the second order field equation in external constant and uniform electromagnetic fields. The pair production probability of vector particles with EDM and AMM is calculated in Sect. 5 with the help of the solutions found. Section 6 is devoted to finding the vacuum polarization of vector particles. Section 7 contains the conclusions.

2 Field equations

One of the non-perturbative approaches of the strong interaction is lattice QCD [20]. For describing fermions on the lattice the Dirac–Kähler equation [21] can be used (see [22]). The Dirac–Kähler's equation in 4-dimensional space-time is given by

$$(\mathrm{d} - \delta + m)\,\Phi = 0,$$

where d is the exterior derivative, $\delta = -\star^{-1}\mathrm{d}\star$ transforms a $n-$form into a $(n-1)$-form, Φ denotes an inhomogeneous differential form. The star operator $\star$ connects a n-form to a $(4-n)$-form so that $\star^2 = 1$, $\mathrm{d}^2 = \delta^2 = 0$. The Laplacian is given by $(\mathrm{d} - \delta)^2 = -(\mathrm{d}\delta + \delta\mathrm{d}) = \partial_\mu\partial_\mu$ where the operator $(\mathrm{d} - \delta)$ is the analog of the Dirac operator $\gamma_\mu\partial_\mu$. The inhomogeneous differential form Φ can be represented as

$$\Phi = \varphi(x) + \varphi_\mu(x)\mathrm{d}x^\mu + \frac{1}{2!}\varphi_{\mu\nu}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \frac{1}{3!}\varphi_{\mu\nu\rho}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho + \frac{1}{4!}\varphi_{\mu\nu\rho\sigma}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma,$$

where $\wedge$ is the exterior product; $\varphi(x)$, $\varphi_\mu(x)$, $\varphi_{\mu\nu}(x)$, $\varphi_{\mu\nu\rho}(x)$, $\varphi_{\mu\nu\rho\sigma}(x)$ are scalar, vector and antisymmetric tensor fields, respectively. The antisymmetric tensors $\varphi_{\mu\nu\rho}(x)$, $\varphi_{\mu\nu\rho\sigma}(x)$ are connected with pseudovector and pseudoscalar fields by the relationships

$$\widetilde{\varphi}_\mu(x) = \frac{1}{3!}\varepsilon_{\mu\nu\rho\sigma}\varphi_{\nu\rho\sigma}(x), \quad \widetilde{\varphi}(x) = \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\varphi_{\mu\nu\rho\sigma}(x),$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the antisymmetric Levy-Civita tensor; $\varepsilon_{1234} = -\mathrm{i}$. The Dirac–Kähler equation formulated in the framework of differential forms [21] is equivalent to the following system of tensor fields [23]:

$$\partial_\nu\psi_{\mu\nu}(x) - \partial_\mu\psi(x) + m^2 B_\mu(x) = 0,$$
$$\partial_\nu\widetilde{\psi}_{\mu\nu}(x) - \partial_\mu\widetilde{\psi}(x) + m^2 C_\mu(x) = 0, \tag{1}$$
$$\partial_\mu B_\mu(x) - \psi(x) = 0, \quad \partial_\mu C_\mu(x) - \widetilde{\psi}(x) = 0, \tag{2}$$
$$\psi_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) - \varepsilon_{\mu\nu\alpha\beta}\partial_\alpha C_\beta(x), \tag{3}$$

where $\widetilde{\psi}_{\mu\nu} = (1/2)\varepsilon_{\mu\nu\alpha\beta}\psi_{\alpha\beta}$ is the dual tensor. Expression (3) is the most general representation for the antisymmetric tensor of second rank [24,25]. Equations (1)–(3) describe the system of the vector $(B_\mu(x))$, pseudovector $(C_\mu(x))$, scalar $(\psi(x))$, and pseudoscalar $(\widetilde{\psi}(x))$ fields. For complex values of the vector potentials $B_\mu(x)$ and $C_\mu(x)$, (1)–(3) correspond to charged vector fields. As the system of (1)–(3) contains two 4-vectors, $B_\mu(x)$, $C_\mu(x)$, which carry spin one and zero (without Lorentz conditions, $\partial_\mu B_\mu(x) \neq 0$, $\partial_\mu C_\mu(x) \neq 0$) there is a doubling of the spin states of the particles. So (1)–(3) describe fields with two spin one and two spin zero states. Using the projection operator technique these states may be separated [23]. The field equations (1)–(3) can be derived from the corresponding Lagrangian and represent the Lagrange–Euler equations. The Proca equations [26] are a special case of (1)–(3) when the constraints $C_\mu = 0$, $\partial_\mu B_\mu = 0$ are imposed. In the case of $C_\mu = 0$, $\partial_\mu B_\mu \neq 0$, we arrive at Stueckelberg's equation [27] describing spin one and zero fields without doubling of the spin states of a particle. The matrix form of (1)–(3) is the 16×16-dimensional Dirac equation [23]. This makes it possible to describe fermions with spin 1/2 with the help of the fields $\psi(x)$, $B_\mu(x)$, $\psi_{\mu\nu}(x)$, $\widetilde{\psi}(x)$, $C_\mu(x)$ which do not realize the tensor representation of the Lorentz group in this case and are connected with spinors. In this work we consider the case when the fields $\psi(x)$, $B_\mu(x)$, $\psi_{\mu\nu}(x)$, $\widetilde{\psi}(x)$, $C_\mu(x)$ are bosonic fields carrying spins 0 and 1.

The Dirac–Kähler equations (1)–(3) are equivalent to the following systems:

$$\begin{aligned}&\partial_\nu M_{\mu\nu}(x)-\partial_\mu M(x)+m^2M_\mu(x)=0,\\&\partial_\mu M_\mu(x)=M(x),\\&M_{\mu\nu}(x)=\partial_\mu M_\nu(x)-\partial_\nu M_\mu(x)-\mathrm{i}\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha M_\beta(x),\end{aligned}\qquad(4)$$

with the self-dual tensor $M_{\mu\nu}(x)=-\mathrm{i}\widetilde{M}_{\mu\nu}(x)$ and

$$\begin{aligned}&\partial_\nu N_{\mu\nu}(x)-\partial_\mu N(x)+m^2N_\mu(x)=0,\\&\partial_\mu N_\mu(x)=N(x),\\&N_{\mu\nu}(x)=\partial_\mu N_\nu(x)-\partial_\nu N_\mu(x)+\mathrm{i}\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha N_\beta(x),\end{aligned}\qquad(5)$$

with the antiself-dual tensor $N_{\mu\nu}(x)=\mathrm{i}\widetilde{N}_{\mu\nu}(x)$, where

$$\begin{aligned}M(x)&=\frac{1}{\sqrt{2}}\left(\psi(x)-\mathrm{i}\widetilde{\psi}(x)\right),\\N(x)&=\frac{1}{\sqrt{2}}\left(\psi(x)+\mathrm{i}\widetilde{\psi}(x)\right),\\M_\mu(x)&=\frac{1}{\sqrt{2}}\left(B_\mu(x)-\mathrm{i}C_\mu(x)\right),\\M_{\mu\nu}(x)&=\frac{1}{\sqrt{2}}\left(\psi_{\mu\nu}(x)-\mathrm{i}\widetilde{\psi}_{\mu\nu}(x)\right),\\N_\mu(x)&=\frac{1}{\sqrt{2}}\left(B_\mu(x)+\mathrm{i}C_\mu(x)\right),\\N_{\mu\nu}(x)&=\frac{1}{\sqrt{2}}\left(\psi_{\mu\nu}(x)+\mathrm{i}\widetilde{\psi}_{\mu\nu}(x)\right).\end{aligned}$$

Adding and subtracting (1)–(3) we get (4) and (5). The self-dual tensor $M_{\mu\nu}$ which obeys (4) is transformed under the $(1,0)$-representation of the Lorentz group and has 3 independent components (see also [28]). Equations (4) are not invariant under the parity transformation and there is no Lagrangian formulation of them. This also applies to (5) for the antiself-dual tensor $N_{\mu\nu}$ which transforms under the $(0,1)$-representation of the Lorentz group. But if we consider the whole system of (4) and (5) (which is equivalent to (1)–(3)) on the basis of the $(0,0)\oplus(1/2,1/2)$ $\oplus\,(1,0)\oplus(0,1)\oplus(1/2,1/2)\oplus(0,0)$-representation of the Lorentz group, we will have a P-invariant theory within the Lagrangian formulation. Each of the system of (4) and (5) describes eight independent variables ($M(x)$, $M_\nu(x)$, $M_{ab}(x)$), ($N(x)$, $N_\nu(x)$, $N_{ab}(x)$).

3 Matrix form of equations

Let us introduce 4-component columns:

$$\xi(x)=-\mathrm{i}m\begin{pmatrix}M_a(x)\\M_4(x)\end{pmatrix},\quad \chi(x)=\begin{pmatrix}\widetilde{M}_a(x)\\M(x)\end{pmatrix},$$
$$\xi'(x)=-\mathrm{i}m\begin{pmatrix}N_a(x)\\N_4(x)\end{pmatrix},\quad \chi'(x)=\begin{pmatrix}\widetilde{N}_a(x)\\N(x)\end{pmatrix},\qquad(6)$$

where $\widetilde{M}_a(x)=(1/2)\varepsilon_{amn}M_{mn}(x)$, $\widetilde{N}_a(x)=(1/2)\varepsilon_{amn}$ $N_{mn}(x)$. Taking into account the notation of (6), (4) and (5) can be represented by

$$\begin{aligned}\alpha_\mu\partial_\mu\xi(x)&=m\chi(x),\\\overline{\alpha}_\mu\partial_\mu\chi(x)&=m\xi(x),\end{aligned}\qquad(7)$$

$$\begin{aligned}\alpha'_\mu\partial_\mu\xi'(x)&=m\chi'(x),\\\overline{\alpha}'_\mu\partial_\mu\chi'(x)&=m\xi'(x),\end{aligned}\qquad(8)$$

where

$$\alpha_1=\begin{pmatrix}0&0&0&-\mathrm{i}\\0&0&-\mathrm{i}&0\\0&\mathrm{i}&0&0\\\mathrm{i}&0&0&0\end{pmatrix},\qquad \alpha_2=\begin{pmatrix}0&0&\mathrm{i}&0\\0&0&0&-\mathrm{i}\\-\mathrm{i}&0&0&0\\0&\mathrm{i}&0&0\end{pmatrix},$$

$$\alpha_3=\begin{pmatrix}0&-\mathrm{i}&0&0\\\mathrm{i}&0&0&0\\0&0&0&-\mathrm{i}\\0&0&\mathrm{i}&0\end{pmatrix},\qquad \alpha'_1=\begin{pmatrix}0&0&0&\mathrm{i}\\0&0&-\mathrm{i}&0\\0&\mathrm{i}&0&0\\\mathrm{i}&0&0&0\end{pmatrix},$$

$$\alpha'_2=\begin{pmatrix}0&0&\mathrm{i}&0\\0&0&0&\mathrm{i}\\-\mathrm{i}&0&0&0\\0&\mathrm{i}&0&0\end{pmatrix},\qquad \alpha'_3=\begin{pmatrix}0&-\mathrm{i}&0&0\\\mathrm{i}&0&0&0\\0&0&0&\mathrm{i}\\0&0&\mathrm{i}&0\end{pmatrix},$$

$$\alpha'_4=\begin{pmatrix}-\mathrm{i}&0&0&0\\0&-\mathrm{i}&0&0\\0&0&-\mathrm{i}&0\\0&0&0&i\end{pmatrix},\qquad \overline{\alpha}'_1=\begin{pmatrix}0&0&0&-\mathrm{i}\\0&0&-\mathrm{i}&0\\0&\mathrm{i}&0&0\\-\mathrm{i}&0&0&0\end{pmatrix},$$

$$\overline{\alpha}'_2=\begin{pmatrix}0&0&\mathrm{i}&0\\0&0&0&-\mathrm{i}\\-\mathrm{i}&0&0&0\\0&-\mathrm{i}&0&0\end{pmatrix},\qquad \overline{\alpha}'_3=\begin{pmatrix}0&-\mathrm{i}&0&0\\\mathrm{i}&0&0&0\\0&0&0&-\mathrm{i}\\0&0&-\mathrm{i}&0\end{pmatrix},$$

$$\overline{\alpha}'_4=\begin{pmatrix}\mathrm{i}&0&0&0\\0&\mathrm{i}&0&0\\0&0&\mathrm{i}&0\\0&0&0&-\mathrm{i}\end{pmatrix},\qquad \alpha_4=\mathrm{i}I_4,\quad \overline{\alpha}_\mu=(\alpha_k,-\mathrm{i}I_4).\qquad(9)$$

Equations (7) and (8) can also be cast in the form

$$\beta_\mu\partial_\mu\varphi(x)+m\varphi(x)=0,\qquad(10)$$
$$\beta'_\mu\partial_\mu\varphi'(x)+m\varphi'(x)=0,\qquad(11)$$

where

$$\varphi(x)=\begin{pmatrix}\chi(x)\\ \xi(x)\end{pmatrix},\quad \beta_\mu=-\begin{pmatrix}0 & \alpha_\mu\\ \overline{\alpha}_\mu & 0\end{pmatrix},$$

$$\varphi'(x)=\begin{pmatrix}\chi'(x)\\ \xi'(x)\end{pmatrix},\quad \beta'_\mu=-\begin{pmatrix}0 & \alpha'_\mu\\ \overline{\alpha}'_\mu & 0\end{pmatrix},\tag{12}$$

and the matrices β_μ, β'_μ obey the Dirac algebra

$$\beta_\mu\beta_\nu+\beta_\nu\beta_\mu=2\delta_{\mu\nu}.\tag{13}$$

We can combine (10) and (11) in the 16-component Dirac-type wave equation, as follows:

$$(\Gamma_\mu\partial_\mu+m)\Psi(x)=0,\tag{14}$$

where

$$\Psi(x)=\begin{pmatrix}\varphi(x)\\ \varphi'(x)\end{pmatrix},\quad \Gamma_\mu=\begin{pmatrix}\beta_\mu & 0\\ 0 & \beta'_\mu\end{pmatrix}.\tag{15}$$

The 16×16-matrices Γ_μ also obey the Dirac algebra (13). This means that the system of (7) and (8) is equivalent to four Dirac equations. So the Dirac–Kähler equations are equivalent to the two matrix equations (10) and (11) (or the two systems of tensor equations (4) and (5)). Equation (10) (and (4)) as well as (11) (and (5)) are parity non-invariant separately and at the same time the system of the two equations (10) and (11) (or the Dirac–Kähler equations) are P-invariant.

Now we will find the symmetry group of (10) and (11). As the matrices β_μ, β'_μ obey the same algebra, (10) and (11) have the same symmetry group. Therefore, we only need to consider (10), which is equivalent to (4).

The matrices β_μ are 8-component Dirac-type matrices, and in a specific basis they take the form $\widehat{\beta}_\mu=I_2\otimes\gamma_\mu$. It is obvious that the matrices $\widehat{\widehat{\beta}}_m=\tau_m\otimes I_4$ (τ_m are the Pauli matrices) form the symmetry algebra of (10). It should be noted that the internal symmetry under consideration is not violated by introducing the electromagnetic fields by the substitution $\partial_\mu\to\partial_\mu-\mathrm{i}eA_\mu$. In the representation (12), the matrices

$$\overline{\beta}_m=\begin{pmatrix}\rho_m & 0\\ 0 & \rho_m\end{pmatrix}\tag{16}$$

commute with the matrices β_μ, if $[\rho_m,\alpha_n]=0$, where the matrices α_n are given by (9) and satisfy the Pauli commutation relations: $\{\alpha_i,\alpha_k\}=2\delta_{ik}$, $[\alpha_i,\alpha_k]=2\mathrm{i}\varepsilon_{ikl}\alpha_l$. Such matrices ρ_m which commute with α_n have the form

$$\rho_1=\begin{pmatrix}0&0&0&-\mathrm{i}\\0&0&\mathrm{i}&0\\0&-\mathrm{i}&0&0\\\mathrm{i}&0&0&0\end{pmatrix},\quad \rho_2=\begin{pmatrix}0&0&-\mathrm{i}&0\\0&0&0&-\mathrm{i}\\\mathrm{i}&0&0&0\\0&\mathrm{i}&0&0\end{pmatrix},$$

$$\rho_3=\begin{pmatrix}0&\mathrm{i}&0&0\\-\mathrm{i}&0&0&0\\0&0&0&-\mathrm{i}\\0&0&\mathrm{i}&0\end{pmatrix};\tag{17}$$

they also obey the Pauli commutation relations. Further we will also use the matrix $\beta_4=\mathrm{i}I_4$.

Let us consider the group of the transformations of the wave function of (10):

$$\varphi(x)\to\exp\left(m_\mu\overline{\beta}_\mu\right)\varphi(x),\tag{18}$$

where the m_μ are four complex parameters. The transformations (18) are defined for the complex fields describing the charged fields; they form the internal symmetry group of (10) which is isomorphic to the $GL(2,c)$ group.

It is possible to apply (10) to the description of spinor particles. In this case the wave function $\varphi(x)$ realizes the spinor representation of the Lorentz group and (10) is equivalent to two Dirac equations; it can be obtained by the variation procedure from the corresponding Lagrangian. Thus, the generators of the Lorentz group are given by

$$J^{(1/2)}_{\mu\nu}=\frac{1}{4}\left(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu\right),\tag{19}$$

and the Hermitianizing matrix is $\eta=\beta_4$.

In the case of the bosonic fields (see (6) and (12)), however, there is no Lagrangian formulation of (10) because it is a parity non-invariant equation based on the reducible $(0,0)\oplus(1/2,1/2)\oplus(1,0)$-representation of the Lorentz group.

The requirement that the Lagrangian of the spinor fields ($\eta=\beta_4$) be invariant under the transformations (18) yields the restriction on the parameters: $m_k^*=-m_k$, $m_4^*=m_4$; this corresponds to the extraction of the $U(2)$ subgroup. According to the Noether theorem, this produces the conservation current

$$\theta_{\mu\alpha}=\overline{\varphi}(x)\beta_\mu\overline{\beta}_\alpha\varphi(x),\tag{20}$$

so that $\partial_\mu\theta_{\mu\alpha}=0$; $\overline{\varphi}(x)=\varphi^+(x)\beta_4$, $\varphi^+(x)$ is the Hermite-conjugate wave function. It is easy to verify that the quantity (20) is also conserved in the boson case (see also [28]), when the fields are given by (6) and (12). We notice that the internal symmetry group of the Dirac–Kähler equation (14) is $GL(4,c)$ and the corresponding Lagrangian for bosonic fields is invariant under the transformations of the $SO(4,2)$ group (or the locally isomorphic group $SU(2,2)$) [29,23].

4 Vector particle with EDM and AMM in uniform electromagnetic field

We consider here the description of electromagnetic interactions of vector particles possessing the EDM and AMM. Sakata and Taketani added some terms in the equations which describe the effects of the anomalous moments [30], and Corben and Schwinger [31] included the AMM in the Proca equations [26]. Yang and Bludman considered an anomalous electric quadrupole moment [32]. Introducing in (4) the interaction with the electromagnetic field $\partial_\mu\to\mathcal{D}_\mu=\partial_\mu-\mathrm{i}eA_\mu$, AMM, and EDM, we arrive at the equations (at the substitutions $M \to \psi$, $M_\mu \to \psi_\mu$, $M_{\mu\nu} \to \psi_{\mu\nu}$)

$$\begin{aligned}
&\mathcal{D}_\mu \psi_\mu = \psi, \\
&\psi_{\mu\nu} = \mathcal{D}_\mu \psi_\nu - \mathcal{D}_\nu \psi_\mu + \sigma\varepsilon_{\mu\nu\alpha\beta}\mathcal{D}_\alpha\psi_\beta, \\
&\mathcal{D}_\nu \psi_{\mu\nu} - \mathcal{D}_\mu \psi + m^2\psi_\mu + \mathrm{i}e\kappa F_{\mu\nu}\psi_\nu = 0. \qquad (21)
\end{aligned}$$

On setting $\sigma = 0$ we get Stueckelberg's equations with the AMM $e\kappa$ which describe fields of spin 1 and 0 [27]. It should be noted that for the field equations (21) the mass of the field with a spin of zero coincides with the mass of the vector field. Quantization of the fields (21) leads to the indefinite metric for a scalar state. At $\sigma = \mathrm{i}$ and $\kappa = 0$ (21) have the 8-component matrix formulation (10) (with the replacement $\partial_\mu \to \mathcal{D}_\mu$) with matrices β_μ (12) obeying the Dirac algebra. It is easy to obtain the second order equation for the 4-vector $\psi_\mu(x)$ from (21). As a result one finds

$$\left(\mathcal{D}_\nu^2 - m^2\right)\psi_\mu(x) + \mathrm{i}e\left(\sigma\widetilde{F}_{\mu\nu} - gF_{\mu\nu}\right)\psi_\nu(x) = 0, \qquad (22)$$

where $\widetilde{F}_{\mu\nu} = (1/2)\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the dual tensor, $g = 1+\kappa$ is the gyromagnetic ratio for the quanta of spin 1. Equation (22) describes a particle with the magnetic moment $eg/(2m)$ and the EDM $\sigma/(2m)$. It should be noted that in the case of the Proca equation with the EDM and AMM, we have in (22) the additional term $(-\mathcal{D}_\mu\mathcal{D}_\nu\psi_\nu)$ due to the absence of a scalar state. Equation (22) can be treated in the framework of the ξ-formalism [33] as a wave equation for a vector field in the gauge $\xi = 1$. We notice that the formal counting of the divergences corresponding to (22) leads to a renormalizable theory due to the form of the field propagator which is proportional to $1/p^2$ but with the presence of an indefinite metric.

It is easier to solve (22) compared to the Proca equation for a particle in external electromagnetic fields. To estimate the physical quantities for a vector particle one needs to eliminate the contribution of a scalar state. In the following calculations we will use this procedure.

Here we will find the solutions of (22) for a particle in the field of uniform and constant electromagnetic fields. We note [13] that the matrices $F_{\mu\nu}$, $\widetilde{F}_{\mu\nu}$ have eigenvalues as follows:

$$\begin{aligned}
&F_{\mu\nu}\psi_\nu^{(\lambda)} = F^{(\lambda)}\psi_\mu^{(\lambda)}, \quad \widetilde{F}_{\mu\nu}\psi_\nu^{(\lambda)} = \frac{1}{F^{(\lambda)}}\mathcal{G}\psi_\mu^{(\lambda)}, \\
&F^{(\lambda)} = \pm F^{(1)}, \pm F^{(2)}, \\
&\lambda = 1,2,3,4, \qquad (23) \\
&F^{(1)} = \frac{\mathrm{i}}{\sqrt{2}}\left[(\mathcal{F}+\mathrm{i}\mathcal{G})^{1/2} + (\mathcal{F}-\mathrm{i}\mathcal{G})^{1/2}\right], \\
&F^{(2)} = \frac{\mathrm{i}}{\sqrt{2}}\left[(\mathcal{F}+\mathrm{i}\mathcal{G})^{1/2} - (\mathcal{F}-\mathrm{i}\mathcal{G})^{1/2}\right], \\
&\mathcal{F} = \frac{1}{4}F_{\mu\nu}^2 = \frac{1}{2}\left(\boldsymbol{H}^2 - \boldsymbol{E}^2\right), \\
&\mathcal{G} = \frac{1}{4}F_{\mu\nu}\widetilde{F}_{\mu\nu} = \boldsymbol{E}\cdot\boldsymbol{H}, \qquad (24)
\end{aligned}$$

and $\boldsymbol{E}$, $\boldsymbol{H}$ are the electric and magnetic fields, respectively. In the diagonal representation (24), (22) becomes

$$\left(D_\nu^2 - m^2\right)\psi_\mu^{(\lambda)}(x) + \mathrm{i}e\left(\sigma\frac{1}{F^{(\lambda)}}\mathcal{G} - gF^{(\lambda)}\right)\psi_\mu^{(\lambda)}(x) = 0. \qquad (25)$$

Equation (25) represents the Klein–Gordon-type equation for every component of the eigenfunction $\psi_\mu^{(\lambda)}(x)$. We consider the general case when the two Lorentz invariants of the electromagnetic fields $\mathcal{F} \neq 0$, $\mathcal{G} \neq 0$. It is convenient to use a coordinate system in which the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ fields are parallel ($\boldsymbol{E} = \boldsymbol{n}E$, $\boldsymbol{H} = \boldsymbol{n}H$, $\boldsymbol{n} = (0,0,1)$) and the 4-vector potential takes the form

$$A_\mu = (0, x_1 H, -tE, 0). \qquad (26)$$

After introducing the variables [34] (see also [35])

$$\begin{aligned}
&\eta = \frac{p_2 - eHx_1}{\sqrt{eH}}, \quad \tau = \sqrt{eE}\left(t + \frac{p_3}{eE}\right), \\
&\psi_\mu^{(\lambda)}(x) = \exp\left[\mathrm{i}\left(p_2x_2 + p_3x_3\right)\right]\Phi_\mu^{(\lambda)}(\eta,\tau), \qquad (27)
\end{aligned}$$

(25) reads

$$\begin{aligned}
&\Big[eH\left(\partial_\eta^2 - \eta^2\right) - eE\left(\partial_\tau^2 + \tau^2\right) \\
&\quad -m^2 + \mathrm{i}e\left(\sigma\frac{1}{F^{(\lambda)}}\mathcal{G} - gF^{(\lambda)}\right)\Big]\Phi_\mu^{(\lambda)}(\eta,\tau) = 0, \qquad (28)
\end{aligned}$$

where $\partial_\eta = \partial/\partial\eta$, $\partial_\tau = \partial/\partial\tau$. The solution to (28) exists in the form

$$\Phi_\mu^{(\lambda)}(\eta,\tau) = \xi_\mu\phi^{(\lambda)}(\eta)\chi^{(\lambda)}(\tau), \qquad (29)$$

with a constant vector ξ_μ, and the eigenfunctions $\phi^{(\lambda)}(\eta)$, $\chi^{(\lambda)}(\tau)$ obey the following equations:

$$\begin{aligned}
&\Big[eH\left(\partial_\eta^2 - \eta^2\right) - m^2 \\
&\quad +\mathrm{i}e\left(\sigma\frac{1}{F^{(\lambda)}}\mathcal{G} - gF^{(\lambda)}\right) + k_\lambda^2\Big]\phi^{(\lambda)}(\eta) = 0, \qquad (30) \\
&\left[eE\left(\partial_\tau^2 + \tau^2\right) + k_\lambda^2\right]\chi^{(\lambda)}(\tau) = 0, \qquad (31)
\end{aligned}$$

where k_λ^2 are the eigenvalues. The finite solution (at $\eta \to \infty$) to (30) is

$$\phi^{(\lambda)}(\eta) = N_0\exp\left(-\frac{\eta^2}{2}\right)H_n(\eta), \qquad (32)$$

where N_0 is the normalization constant, and $H_n(\eta)$ are Hermite polynomials. The requirement that this solution be finite leads to the condition

$$\begin{aligned}
&k_\lambda^2 - m^2 + \mathrm{i}e\left(\sigma\frac{1}{F^{(\lambda)}}\mathcal{G} - gF^{(\lambda)}\right) = eH(2n+1), \\
&\qquad n = 1,2,..., \qquad (33)
\end{aligned}$$

n is the principal quantum number and k_λ is the spectral parameter. Equation (31) has four solutions with different asymptotics at $t\to\pm\infty$ [34]. We have

$$
\begin{aligned}
{}_{+}\chi^{(\lambda)}(\tau) &= D_\nu[-(1-\mathrm{i})\tau],\\
{}^{-}\chi^{(\lambda)}(\tau) &= D_\nu[(1-\mathrm{i})\tau],\\
{}^{+}\chi^{(\lambda)}(\tau) &= D_{\nu^*}[(1+\mathrm{i})\tau],\\
{}_{-}\chi^{(\lambda)}(\tau) &= D_{\nu^*}[-(1+\mathrm{i})\tau],
\end{aligned}
\tag{34}
$$

where $D_\nu(x)$ are the parabolic-cylinder functions (the Weber–Hermite functions) and

$$\nu=\frac{\mathrm{i}k_\lambda^2}{2eE}-\frac{1}{2}.$$

The four solutions of (25) for the potential (26) with different asymptotic forms are given by

$$
\begin{aligned}
{}^{\pm}_{\pm}\psi^{(\lambda)}_\mu(x) = N_0\xi_\mu \exp\left\{\mathrm{i}(p_2x_2+p_3x_3)-\frac{\eta^2}{2}\right\}\\
\times H_n(\eta){}^{\pm}_{\pm}\chi^{(\lambda)}(\tau).
\end{aligned}
\tag{35}
$$

The exact solutions (35) will be used to estimate the pair production probability of vector particles and antiparticles in the external constant and uniform electromagnetic fields.

5 Pair production of vector particles with EDM and AMM

The probability for pair production of vector particles with the EDM and AMM by constant electromagnetic fields can be obtained through the asymptotic form of solutions (35) when the time $t\to\pm\infty$. The functions ${}^{+}_{+}\psi^{(\lambda)}(\tau)$ at $t\to\pm\infty$ have a positive frequency and ${}^{-}_{-}\psi^{(\lambda)}(\tau)$ have a negative frequency. The three quantities k_λ^2 and the momentum projections p_2, p_3 entering solutions (34) and (35) are conserved. The functions (34) (see [34]) obey the relations

$$
\begin{aligned}
{}_{+}\chi^{(\lambda)}(\tau) &= c_{1n\lambda}{}^{+}\chi^{(\lambda)}(\tau)+c_{2n\lambda}{}^{-}\chi^{(\lambda)}(\tau),\\
{}^{+}\chi^{(\lambda)}(\tau) &= c^*_{1n\lambda}{}_{+}\chi^{(\lambda)}(\tau)-c_{2n\lambda}{}_{-}\chi^{(\lambda)}(\tau),\\
{}^{-}\chi^{(\lambda)}(\tau) &= -c^*_{2n\lambda}{}_{+}\chi^{(\lambda)}(\tau)+c_{1n\lambda}{}_{-}\chi^{(\lambda)}(\tau),\\
{}_{-}\chi^{(\lambda)}(\tau) &= c^*_{2n\lambda}{}^{+}\chi^{(\lambda)}(\tau)+c^*_{1n\lambda}{}^{-}\chi^{(\lambda)}(\tau),
\end{aligned}
\tag{36}
$$

where

$$
\begin{aligned}
c_{2n\lambda} &= \exp\left[-\frac{\pi}{2}(\varepsilon+\mathrm{i})\right],\\
\varepsilon &= \frac{m^2-\mathrm{i}e\left(\sigma\frac{1}{F^{(\lambda)}}\mathcal{G}-gF^{(\lambda)}\right)+eH(2n+1)}{eE},\\
&\mid c_{1n\lambda}\mid^2-\mid c_{2n\lambda}\mid^2=1.
\end{aligned}
\tag{37}
$$

The quantity $c_{2n\lambda}$ allows us to calculate the probability of pair production of vector particles in the state with the quantum number n and corresponding to the eigenvalue $F^{(\lambda)}$. The probability for the production of a pair of vector particles in the state with quantum number n, components of momentum p_2, p_3 and corresponding to the eigenvalue $F^{(\lambda)}$ throughout all space and during all time, is

$$
\begin{aligned}
\mid c_{2n\lambda}\mid^2 = \exp\left\{-\pi\left[\frac{m^2}{eE}+\frac{H}{E}(2n+1)\right]\right\}\\
\times\left|\exp\left[\mathrm{i}\pi\left(\sigma\frac{1}{F^{(\lambda)}}\mathcal{G}-gF^{(\lambda)}\right)/E\right]\right|.
\end{aligned}
\tag{38}
$$

The expression (38) also gives the probability of the annihilation of a pair of particles with quantum numbers n, p_2, p_3. From (38) we find the average number of pairs produced from a vacuum

$$\overline{N}=\int\sum_{n,\lambda}\mid c_{2n\lambda}\mid^2\,\mathrm{d}p_2\mathrm{d}p_3\frac{L^2}{(2\pi)^2},\tag{39}$$

where $(2\pi)^{-2}\mathrm{d}p_2\mathrm{d}p_3L^2$ means the final state density with the cut-off L along the coordinates ($V=L^3$ is the normalization volume). In accordance with the approach [34] we can use the substitutions

$$\int\mathrm{d}p_2\to eHL,\qquad\int\mathrm{d}p_3\to eET.\tag{40}$$

Here T is the time of observation. It is possible to calculate the sum in (39) over the principal quantum number n, and the eigenvalues λ with the help of (38) and (24). Using (40) we obtain the probability of pair production of particles per unit volume and per unit time

$$
\begin{aligned}
I(E,H)=\frac{\overline{N}}{VT}=\frac{e^2EH}{8\pi^2}\frac{\exp\left[-\pi m^2/(eE)\right]}{\sinh(\pi H/E)}\\
\times\sum_\lambda\left|\exp\left[\mathrm{i}\pi\left(\sigma\frac{1}{F^{(\lambda)}}\mathcal{G}-gF^{(\lambda)}\right)/E\right]\right|.
\end{aligned}
\tag{41}
$$

Evaluating the sum with the help of (24) we find

$$
\begin{aligned}
\sum_\lambda\left|\exp\left[\pi\left(\sigma\frac{1}{F^{(\lambda)}}\mathcal{G}-\mathrm{i}gF^{(\lambda)}\right)/E\right]\right|\\
=2\cosh\pi\left(\sigma+g\frac{H}{E}\right)+2,
\end{aligned}
\tag{42}
$$

we arrive at the pair production probability

$$
\begin{aligned}
I(E,H)=\frac{e^2EH}{4\pi^2}\frac{\cosh\pi(\sigma+gH/E)+1}{\sinh(\pi H/E)}\\
\times\exp\left[-\pi m^2/(eE)\right].
\end{aligned}
\tag{43}
$$

So $I(E,H)$ is the intensity of the creation of pairs of particles with a spin of 1, 0. Below we extract the pair production probability for particles with the pure spin 1 possessing the gyromagnetic ratio g (and magnetic moment $\mu=eg/(2m)$) and the EDM $\sigma/(2m)$.

It follows from (43) that there is pair production in a purely magnetic field if $g>1$, which is an indication of the instability of the vacuum in the magnetic field. For the case $g > 1$, $\sigma = 0$, this property for the higher spin particles was pointed out in [16].

It is interesting to compare the probability (43) with those for particles possessing pure spin 1. The probability of pair production per unit volume and per unit time of vector particles on the base of the $(0,1) \oplus (1,0)$-representation of the Lorentz group at $\sigma = 0$ is given by [16, 17]

$$I^{(1)}(E,H) = \frac{e^2EH}{8\pi^2}\frac{\exp\left[-\pi m^2/(eE)\right]}{\sinh(\pi H/E)}\frac{\sinh\left[3g\pi H/(2E)\right]}{\sinh\left[g\pi H/(2E)\right]}. \tag{44}$$

Setting $\sigma = 0$ in (43) and using some transformations we arrive at the equality

$$I(E,H) = I^{(1)}(E,H) + I^{(0)}(E,H), \tag{45}$$

where

$$I^{(0)}(E,H) = \frac{e^2EH}{8\pi^2}\frac{\exp\left[-\pi m^2/(eE)\right]}{\sinh(\pi H/E)}$$

is the intensity of the creation of pairs of scalar particles [13] (see also the creation of pairs of composite scalar particles in [36]). The physical meaning of (45) is clear: the probability of pair production of fields with spin 1, 0 is the sum of the production probabilities of vector and scalar particles. By excepting (45) for arbitrary σ and g we obtain from (43) the expression for the pair production probability of particles with pure spin one:

$$I^{(1)}(E,H) = \frac{e^2EH}{8\pi^2}\frac{2\cosh\pi\left(\sigma + gH/E\right) + 1}{\sinh(\pi H/E)} \times \exp\left[-\pi m^2/(eE)\right]. \tag{46}$$

Equation (46), the one that we obtained, is a new result for the intensity of pair production of vector particles with the EDM and AMM. From the general formula (46) we find that in the case $\sigma = g = 0$, pair production of vector particles is three times that for scalar pair production. This is due to the three physical degrees of freedom of the vector field.

The imaginary part of the density of the Lagrangian can be obtained using the relationship [34]

$$VT\mathrm{Im}\mathcal{L} = \frac{1}{2}\int\sum_{n,\lambda}\ln\mid c_{1n\lambda}\mid^2 \mathrm{d}p_2\mathrm{d}p_3\frac{L^2}{(2\pi)^2}. \tag{47}$$

From (47), taking into account (37) and (40), we arrive at

$$\mathrm{Im}\mathcal{L} = \frac{e^2EH}{8\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\exp\left(-\frac{\pi k m^2}{eE}\right) \times\frac{\cosh\pi k\left(\sigma + gH/E\right) + 1}{\sinh(\pi k H/E)}. \tag{48}$$

According to the approach of [13] the first term in (48) (at $k = 1$) coincides with the intensity of pair production (43) (probability of the pair production per unit volume per unit time) divided by 2. The expression $\mathrm{Im}\mathcal{L}$ (48) and the pair production probability (46) do not depend on the renormalization scheme, because all divergences and the renormalizability are contained in $\mathrm{Re}\mathcal{L}$ [13].

6 Polarization of vector particle vacuum

In this section we evaluate one-loop corrections to the Lagrange function of a constant and uniform electromagnetic field due to the field interaction with a vacuum of vector particles with the EDM and AMM. This problem has been solved for a number of theories [13,15–18]. The effect of scattering of light by light is described by the non-linear corrections to the Lagrangian of the electromagnetic field. Adapting the Schwinger method [13] to the fields described by (22), we obtain the non-linear corrections to the Lagrangian of a constant and uniform electromagnetic field:

$$\mathcal{L}^{(1)} = \frac{1}{16\pi^2}\int_0^\infty \mathrm{d}\tau\tau^{-3}\exp\left(-m^2\tau - l(\tau)\right) \times\mathrm{tr}\exp\left[\mathrm{i}e_0\left(\sigma\widetilde{F}_{\mu\nu} - gF_{\mu\nu}\right)\tau\right], \tag{49}$$

with

$$l(\tau) = \frac{1}{2}\mathrm{tr}\ln\left[(e_0F\tau)^{-1}\sin(e_0F\tau)\right],$$
$$\exp\left[-l(\tau)\right] = \frac{(e_0\tau)^2\mathcal{G}_0}{\mathrm{Im}\cosh(e_0\tau X_0)}, \tag{50}$$

where $\boldsymbol{X}_0 = \boldsymbol{H}_0 + \mathrm{i}\boldsymbol{E}_0$, $X = (\boldsymbol{X}^2)^{1/2}$, $\mathcal{G}_0 = \boldsymbol{E}_0\boldsymbol{H}_0$; $\boldsymbol{E}_0$, $\boldsymbol{H}_0$ are bare (non-renormalized) electric and magnetic fields, respectively, e_0 is the bare electric charge (the index 0 refers to the unrenormalized variables). The expression (49) is the effective non-linear Lagrangian which is represented as an integral over the proper time τ. Here we consider the general case of arbitrary constant vectors $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$. With the help of (24) we calculate the trace (tr) of the matrices, finding

$$\mathrm{tr}\exp\left[\mathrm{i}e_0\left(\sigma\widetilde{F}_{\mu\nu} - gF_{\mu\nu}\right)\tau\right] = 2\left[\cosh e_0\tau\left(\frac{\sigma\mathcal{G}_0}{\mathrm{Re}X_0} + g\mathrm{Re}X_0\right) + \cos e_0\tau\left(\frac{\sigma\mathcal{G}_0}{\mathrm{Im}X_0} - g\mathrm{Im}X_0\right)\right]. \tag{51}$$

Substituting (51) into (49) and subtracting the additive constant to ensure that the expression $\mathcal{L}^{(1)}$ vanishes for zero fields, we get

$$\mathcal{L}^{(1)} = \frac{1}{8\pi^2}\int_0^\infty d\tau^{-3}\exp\left(-m^2\tau\right) \times\Big[(e_0\tau)^2\mathcal{G}_0(\cosh e_0\tau\left(\sigma\mathcal{G}_0/\mathrm{Re}X_0 + g\mathrm{Re}X_0\right) + \cos e_0\tau\left(\sigma\mathcal{G}_0/\mathrm{Im}X_0 - g\mathrm{I\ m}X_0\right)) \Big/ (\mathrm{Im}\cosh(e_0\tau X_0)) - 2\Big], \tag{52}$$

$$\begin{aligned}\mathcal{L} &= \mathcal{L}^{(0)} + \mathcal{L}^{(1)} \\ &= -\mathcal{F} + \frac{1}{8\pi^2}\int_0^\infty \mathrm{d}\tau\tau^{-3}\exp\left(-m^2\tau\right) \\ &\times\left[(e\tau)^2\mathcal{G}\frac{\cosh e\tau\left(\sigma\mathcal{G}/\mathrm{Re}X + g\mathrm{Re}X\right) + \cos e\tau\left(\sigma\mathcal{G}/\mathrm{Im}X - g\mathrm{Im}X\right)}{\mathrm{Im}\cosh(e\tau X)} - 2 + (e\tau)^2\left(\frac{2}{3}+\sigma^2-g^2\right)\mathcal{F}\right], \end{aligned} \tag{54}$$

The integral (52) is the non-linear correction to Maxwell's Lagrangian due to the vacuum polarization of vector (with the additional scalar field) fields which possess the EDM and AMM. The Lagrangian (52) contains the term that renormalizes the Lagrangian of the free electromagnetic fields

$$\mathcal{L}^{(0)} = -\mathcal{F}_0 = \frac{1}{2}\left(\boldsymbol{E}_0^2 - \boldsymbol{H}_0^2\right). \tag{53}$$

Extracting the divergent constant in (52) for weak fields, and adding (52) to the Maxwell Lagrangian (53), we obtain the renormalized Lagrangian of electromagnetic fields (see (54) on top of the page) where the renormalized fields and charges are used:

$$\mathcal{F} = Z_3^{-1}\mathcal{F}_0, \quad e = Z_3^{1/2}e_0,$$

and the renormalization constant is given by

$$Z_3^{-1} = 1 + \frac{e_0^2}{12\pi^2}\left[1+\frac{3}{2}\left(\sigma^2-g^2\right)\right] \times\int_0^\infty \mathrm{d}\tau\tau^{-1}\exp\left(-m^2\tau\right). \tag{55}$$

The integral (54) vanishes already if the electromagnetic fields $\boldsymbol{E}$, $\boldsymbol{H}$ are absent. We can use the cutoff factor τ_0 at the lower limit in the integral (55), and the constant Z_3^{-1} diverges logarithmically as $\tau_0 \to 0$. When the EDM is absent ($\sigma = 0$), and the gyromagnetic ratio $g = 2$, that is, in the linear approximation to the renormalizable gauge theory, we arrive by (55) at the renormalization constant obtained in [15]. It follows from (55) that when the inequality

$$g^2 - \sigma^2 > \frac{2}{3} \tag{56}$$

is valid, the renormalization constant of the charge $Z_3^{1/2}$ becomes larger than one. This case, unlike ordinary electrodynamics, corresponds to the absence of the zero charge situation in the asymptotic region and indicates asymptotic freedom in the field [37,38]. According to (56) the asymptotically free behavior in the vector field is due to the AMM, but the role of the EDM is opposite. In the case $\sigma^2 - g^2 > 2/3$ the situation of the zero charge situation in the asymptotic region, like electrodynamics, is realized. From (55) we obtain the Callan–Zymanzik β-function that corresponds to the renormalizable theory

$$\beta = \frac{e_0^2}{12\pi^2}\left[1+\frac{3}{2}\left(\sigma^2-g^2\right)\right]. \tag{57}$$

Under the condition (56) the β-function is negative ($\beta < 0$) and we arrive at the region of asymptotic freedom. The AMM ensures asymptotic freedom and instability of the vacuum in a magnetic field.

Expanding (54) in small electromagnetic fields we obtain the Maxwell Lagrangian including the non-linear corrections (in rational units)

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{H}^2\right) + \frac{6\sigma g}{2+3(\sigma^2-g^2)}\left(\mathcal{G}-\mathcal{G}_0\right) \\ &+\frac{\alpha^2}{m^4}\left[\frac{14-30(g^2-\sigma^2)+15(g^4+\sigma^4)}{45}\mathcal{F}^2\right. \\ &+\frac{2}{45}\left(1+\frac{15(\sigma^4+6\sigma^2g^2+g^4)}{4}\right)\mathcal{G}^2 \\ &\left.+\frac{2}{3}\sigma g\left(g^2-\sigma^2-2\right)\mathcal{G}\mathcal{F}\right], \end{aligned} \tag{58}$$

where $\alpha = e^2/(4\pi)$. The second term in (58) is the induced parity violation anomaly for a vector field with the EDM. This and the last terms in (58) violate parity symmetry due to the EDM of a particle. The effective Lagrangian (58) is like the Heisenberg–Euler Lagrangian of QED [39, 40], but in the case of the polarized vacuum of vector fields with arbitrary EDM and AMM and an additional scalar field (with the same mass). The presence of a scalar field is due to the special gauge $\xi = 1$ which was chosen to simplify the calculations. Now we will take into consideration the contribution of a scalar (non-physical) field. It is easy to verify that for the particular case of $\sigma = 0$, $g = 0$, (58) becomes

$$\begin{aligned}\mathcal{L}(\sigma = g = 0) &= \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{H}^2\right) \\ &+\frac{\alpha^2}{90m^4}\left[7\left(\boldsymbol{E}^2-\boldsymbol{H}^2\right)^2 + 4(\boldsymbol{E}\boldsymbol{H})^2\right] \\ &= \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{H}^2\right) + 4\mathcal{L}_{\text{spin }0}, \end{aligned} \tag{59}$$

where

$$\mathcal{L}_{\text{spin }0} = \frac{\alpha^2}{360m^4}\left[7\left(\boldsymbol{E}^2-\boldsymbol{H}^2\right)^2 + 4(\boldsymbol{E}\boldsymbol{H})^2\right] \tag{60}$$

is the correction to the Maxwell Lagrangian due to the vacuum polarization of scalar point-like particles [13]. As (22) becomes a Klein–Gordon equation for the field ψ_μ at $\sigma = g = 0$, there is an equal contribution of four degrees of freedom of fields with spins 1 (three projections ± 1, 0) and 0. To have the contribution from a field of pure spin

1 we should subtract from (58) the expression (60) corresponding to spin 0 of a field. As a result the Lagrangian of a constant, uniform, electromagnetic field taking into account the vacuum polarization of a charged vector particles with the EDM and AMM is given by

$$\begin{aligned}\mathcal{L}_{\text{spin }1} &= \mathcal{L} - \mathcal{L}_{\text{spin }0} = \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{H}^2\right)\\ &+ \frac{6\sigma g}{2+3(\sigma^2-g^2)}\left(\boldsymbol{EH} - \boldsymbol{E}_0\boldsymbol{H}_0\right)\\ &+ \frac{\alpha^2}{m^4}\left[\frac{7-20(g^2-\sigma^2)+10(g^4+\sigma^4)}{120}\left(\boldsymbol{E}^2-\boldsymbol{H}^2\right)^2\right.\\ &+ \frac{1+5(\sigma^4+6\sigma^2g^2+g^4)}{30}(\boldsymbol{EH})^2\\ &\left.+\frac{1}{3}\sigma g\left(g^2-\sigma^2-2\right)(\boldsymbol{EH})\left(\boldsymbol{E}^2-\boldsymbol{H}^2\right)\right]. \qquad (61)\end{aligned}$$

For the particular case $\sigma = 0$, $g = 2$, which corresponds to the linear approximation to the renormalizable SM, (61) leads to the expression

$$\begin{aligned}\mathcal{L}_{\text{spin }1} &= \frac{1}{2}\left(\boldsymbol{E}^2-\boldsymbol{H}^2\right) + \frac{\alpha^2}{10m^4}\\ &\times\left[\frac{29}{4}\left(\boldsymbol{E}^2-\boldsymbol{H}^2\right)^2 + 27(\boldsymbol{EH})^2\right], \qquad (62)\end{aligned}$$

which coincides with those obtained in [15].

It is possible to obtain the asymptotic form of (54) for super-critical fields at $eE/m^2 \to \infty$ and $eH/m^2 \to \infty$. However, for strong electromagnetic fields the AMM and EDM can depend on the external field like the dependence of the electron AMM in QED [41,42].

7 Conclusion

Starting with the Dirac–Kähler equation for tensor fields we arrived at the two P-odd subsystem for self-dual and antiself-dual antisymmetric tensors of second rank. These equations are based on the $(0,0)\oplus(1/2,1/2)\oplus(1,0)$ and $(0,1)\oplus(1/2,1/2)\oplus(0,0)$-representations of the Lorentz group and describe fields with spins of 1 and 0. The 8-component Dirac-like P-odd matrix wave equation is constructed; it possesses the $GL(2,c)$ group of symmetry. This symmetry is due to the presence of two spins, 1 and 0. The system of tensor equations considered allows us to introduce the EDM and AMM of a particle in the first order equations. The second order equation for a particle with the EDM and AMM is simpler (for solving) compared to the Proca equation. This equation can be treated as an equation for a vector particle with the gauge $\xi = 1$ in the framework of the Lee and Yang formalism. The contribution of the non-physical scalar field to physical observables is eliminated at the end of the calculations. Such an approach allowed us to obtain the pair production probability, and the effective Lagrangian for electromagnetic fields taking into account the polarization of the vacuum of vector particles with the EDM and AMM. This is the generalization of the Schwinger result for the case of vector particles in external electric and magnetic fields. The exact formula for the intensity of pair production of fields with spin 1 and 0 is the sum of the intensity of pair production of vector and scalar particles. It is shown that there is pair production of vector particles by a purely magnetic field in the case of $g > 1$ ensuring asymptotic freedom and instability of the vacuum in a magnetic field. The role of the EDM of a vector particle is opposite: the EDM of a particle does not lead to instability of the vacuum in a magnetic field and suppresses the phenomenon of asymptotic freedom. The pair production probability does not depend on the renormalization scheme because all divergences and the renormalizability are contained in Re$\mathcal{L}$. Discussing the procedure of the renormalization we imply that the scheme considered is the linearized version of renormalized gauge theory. This point of view is due to the smallness of the vector field self-interaction constant (see [14]), and it is possible to ignore processes that allow for the self-interaction of the vector field in vacuum.

The presence of the EDM and the value of the AMM $\kappa \neq 1$ ($g \neq 2$) of a vector particle leads to physics beyond the SM. Recent experimental muon AMM data [43] have challenged the SM as there is a discrepancy of 2.6σ deviation between the theory and the averaged experimental value. This can open a window to new physics.

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